MA1011 - NUMERICAL METHODS

UNIT - II

INTERPOLATION AND APPROXIMATION

PART - A

1. **Interpolation:**

We are given a table values of X and Y. The process of finding the value of y (not given in the table) corresponding to the value of x is known as "interpolation".

Inverse Interpolation:

The process of finding the value of x (not given in the table) corresponding to the value of y is known as "inverse interpolation".

2. Write Newton's forward and backward difference formula where they are used?

Newton's Forward Formula:

$$y(x) = y_0 + \frac{u\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots + \frac{u(u-1)(u-2)\dots(u-\overline{n-1})}{n!}\Delta^n y_0.$$

Where $u = \frac{x-x_0}{h}$

It is used to find the value of y for the value of x near the beginning of the table value.

Newton's Backward Formula:

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \dots + \frac{v(v+1)\dots(v+n-1)}{n!}\nabla^n y_n.$$

Where $v = \frac{x-x_n}{h}$

It is used to find the value of y for the value of x at the end of the table value.

3. Obtain the interpolation Quadrate polynomial for the given data by using Newtons forward difference formula.

X :	0	2	4	6
Y :	-3	5	21	45

Table Calculation:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-3			
		8		
2	5		8	
		16		0
4	21		8	
		24		
6	45			

$u = \frac{x-x_0}{h}$, $h = 2-0 = 2$

$u = \frac{x-0}{2} = x/2$

Formula:

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$$

$$\Rightarrow -3 + \frac{x}{2}\times 8 + \frac{x/2\,(x/2 - 1)}{2}\times 8$$

$$\Rightarrow -3 + 4x + x^2 - 2x$$

$$\Rightarrow x^2 + 2x - 3$$

4. Using Newtons backward difference formula. Write the formula for the first and the second order derivatives at the end value $x = x_n$ upto the fourth order difference term.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right]$$

5. State Newtons divided difference formula.

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)\dots(x-x_{n-1})f(x_0,x_1,\dots,x_n)$$

6. State the properties of divided differences.

The divided differences are symmetrical in all their arguments.

1. $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$ (∵ Linear Property)
2. $\Delta\, c f(x) = c\, \Delta f(x)$
3. The divided difference of n^{th} degree polynomial is constant.

7. Show that $\Delta^3_{bcd}(1/a) = -\frac{1}{abcd}$

Given:

$\Delta^3_{bcd}(1/a) = -1/abcd$ Let us take $f(x) = 1/x$; $f(a) = 1/a$.

Solution

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$1/a$	$\frac{1/b - 1/a}{b-a} = -1/ab$	$\frac{-1/cb + 1/ab}{c-a} = 1/abc$	$\frac{1/abc - 1/bcd}{a-d} = -\frac{1}{abcd}$
b	$1/b$	$\frac{1/c - 1/b}{c-b} = -1/cb$	$\frac{-1/cd + 1/cb}{d-b} = 1/bcd$	
c	$1/c$	$\frac{1/d - 1/c}{d-c} = -1/cd$		
d	$1/d$			

8. Find the Second divided differences arguments a, b, c if $f(x) = 1/x$.

$$\underset{b}{\Delta}\,(1/a) = f(a,b)$$

$$= \frac{f(b) - f(a)}{b-a}$$

$$= \frac{1/b - 1/a}{b-a} \Rightarrow \frac{a-b}{ab(b-a)}$$

$$f(a,b) = -1/ab$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a}$$

$$= \frac{-1/bc + 1/ab}{c-a}$$

$$= \frac{(c-a)}{abc(c-a)} = 1/abc$$

9. If $f(x) = \frac{1}{x^2}$ with arguments a, b find $f(a,b)$

$$f(a,b) = \frac{f(b) - f(a)}{b-a}$$

$$= \frac{1/b^2 - 1/a^2}{b-a}$$

$$= \frac{a^2 - b^2}{a^2b^2(b-a)} = \frac{(a+b)(a-b)}{a^2b^2(b-a)}$$

$$= -\frac{(a+b)}{a^2b^2}.$$

10. Find the 3rd divided differences of $f(x)$ given with the arguments 2, 4, 9, 10 Where $f(x) = x^3 - 2x$.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	6	$\frac{56-6}{4-2} = 25$	$\frac{131-25}{9-2} = 15.14$	$\frac{189.66-15.14}{10-2} = 21.815$
4	56	$\frac{711-56}{9-4} = 131$	$\frac{1269-131}{10-4} = 189.66$	
9	711	$\frac{1980-711}{10-9} = 1269$		
10	1980			

11. Write Lagrange's Interpolation formula and inverse formula.

Lagrange's Interpolation Formula:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} y_1$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{\ldots\ldots\ldots} y_n$$

Inverse Lagrange's Interpolation Formula:

$$x = \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)} x_0$$

$$+ \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)} x_1$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})} x_n$$

12. Compare Newtons formula with Lagrange's formula.

No	Newtons formula	Lagrange's formula
1.	Applicable only for the equal intervals	Applicable both for the equal and unequal intervals.
2.	The differences of the dependent Variable should be smaller.	The difference of this dependent Variable is smaller (or) not the formula can be Used.

13. Find the divided difference for the following data.

X :	2	5	10
Y :	5	29	109

X	Y	$\Delta f(x)$	$\Delta^2 f(x)$
2	5	$\frac{29-5}{5-2} = 8$	$\frac{16-8}{10-2} = 1$
5	29	$\frac{109-29}{10-5} = 16$	
10	109		

14. Given $f(0) = -2$, $f(1) = 2$, $f(2) = 8$. Find the root of $f(x) = 0$ Using Newtons interpolation formula.

X	Y	ΔY	$\Delta^2 Y$
0	-2	4	2
1	2	6	
2	8		

$$u = \frac{x - x_0}{h} \; ; \; = \frac{x-0}{1} \quad \therefore \boxed{u = x}$$

Formula:

$$y(x) = y_0 + \frac{u}{1!}\Delta y + \frac{u(u-1)}{2!}\Delta^2 y$$

$$= -2 + x\,4 + \frac{x(x-1)}{2}\,2$$

$$= -2 + x \cdot 4 + \frac{x(x-1)}{2} 2$$

$$= -2 + 4x + x^2 - x$$

$$y = x^2 + 3x - 2$$

$$y(x) = 0 \Rightarrow x^2 + 3x - 2 = 0$$

$$x = \frac{-3 \pm \sqrt{17}}{2}$$

15. Write the formula to find Cubic Spline.

Let $S(x)$ is a cubic Polynomial. $S''(x)$ is linear in each interval (x_{i-1}, x_i).

$$\boxed{\begin{aligned} S(x) = {} & \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] \\ & + \frac{1}{h}\left[(x_i - x)\left(y_{i-1} - \frac{h^2}{6} M_{i-1}\right)\right] \\ & + \frac{1}{h}\left[(x - x_{i-1})\left(y_i - \frac{h^2}{6} M_i\right)\right] \end{aligned}}$$

Where,

$$\boxed{M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]}$$

$i = 1, 2, \ldots n-1.$

PART- B

1. From the following data estimate the number of persons earning weekly wages between 60 and 70 rupees.

Wages in Rs : Below	40	40-60	60-80	80-100	100-120
No. of Persons	250	120	100	70	50.

Solution:

Table Calculation:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		10	
100	540		-20		
		50			
120	590				

To find the no. of Persons Earning less than Rs.70.

$$u = \frac{x - x_0}{h} = \frac{70-40}{20} = 1.5$$

$$\boxed{u = 1.5}$$

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$$y(70) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots$$

$$= 250 + 1.5(120) + \frac{1.5(1.5-1)}{2}(-20) + \ldots\ldots$$

$$= 423.59.$$

$$\boxed{y(70) = 424}$$

No. of Persons Earning less than Rs. 60 is 370.

No. of Persons Earning between 70 and 60 is

$$\Rightarrow 424 - 370$$

$$= 54.$$

2. The population of a town is as follows

Year X :	1941	1951	1961	1971	1981	1991
Population Y :	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976.

Solution:

Table Calculation:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	$\boxed{20}$					
		$\boxed{4}$				
1951	24		$\boxed{1}$			
		5		$\boxed{1}$		
1961	29		2		$\boxed{0}$	
		7		1		$\boxed{1}$
1971	36		3		$\boxed{1}$	
		10		$\boxed{2}$		
1981	46		$\boxed{5}$			
		$\boxed{5}$				
1991	$\boxed{51}$					

To estimate the population during 1946 use Newton's forward formula and for 1976 we use backward formula.

Formula:

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y.$$

$$u = \frac{x - x_0}{h} = \frac{1946-1941}{10} = \frac{5}{10} = 0.5$$

$$\boxed{u = 0.5}$$

$$\boxed{y(1976) = 40.809}$$

$\therefore$ Increase in population during the Period 1946 to 1976 is $\Rightarrow 40.809 - 21.69$

$\Rightarrow 19.119$ lakhs.

3. Obtain the root of $f(x)=0$ by Lagrange's inverse interpolation given that $f(30) = -30$; $f(34) = -13$, $f(38) = 3$, $f(42) = 18$.

x :	30	34	38	42
$y = f(x)$:	-30	-13	3	18

To find the root, find x when $y=0$.

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3.$$

$$y(1946) = 20 + 0.5(4) + \frac{0.5(0.5-1)}{2}1 + \frac{0.5(0.5-1)(0.5-2)}{6}1 + 0$$

$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{5.4.3.2.1} \times 1$$

$$\Rightarrow 20 + 2 + \frac{0.5(-0.5)}{2} + \frac{0.5(-0.5)(-1.5)}{6}$$

$$+ \frac{0.5(-0.5)(-1.5)(-2.5)(-3.5)}{120}.$$

$$\boxed{y(1946) = 21.69}$$

To find $y(1976)$:

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n +$$

$$\frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \frac{v(v+1)(v+2)(v+3)(v+4)}{5!}\Delta^5 y_n$$

$$v = \frac{x-x_n}{h} = \frac{1976-1991}{10} = -\frac{15}{10} = -1.5$$

$$\boxed{v = -1.5}$$

$$y(1976) = 51 + (-1.5)5 + \frac{(-1.5)(-1.5+1)}{2}5 + \frac{(-1.5)(-1.5+1)(-1.5}{6}$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{24}(1) + \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+}{120}.$$

Put $y=0$

$$x = \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13-0)(-13-3)(-13-18)}(34)$$

$$+ \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$

$$\boxed{x = 37.23}$$

4. Using Newtons forward interpolation formula find the Cubic polynomial which takes the following Values.

x :	0	1	2	3
f(x) :	1	2	1	10

Evaluate $f(4)$ using Newtons backward formula.

Solution:

Here $x = 4$

$$v = \frac{x-x_n}{h} = \frac{4-3}{1} = 1 \qquad \therefore \boxed{v=1}$$

Formula:

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \ldots$$

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Put $y=0$

$$x = \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13-0)(-13-3)(-13-18)}(34)$$

$$+ \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$

$$\boxed{x = 37.23}$$

4. Using Newtons forward interpolation formula find the Cubic polynomial which takes the following Values.

x :	0	1	2	3
f(x) :	1	2	1	10

Evaluate $f(4)$ using Newtons backward formula.

Solution:

Here $x = 4$

$$v = \frac{x-x_n}{h} = \frac{4-3}{1} = 1 \qquad \therefore \boxed{v=1}$$

Formula:

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \ldots$$

x	y	Δy	Δ²y	Δ³y
0	1			
		1		
1	2		-2	
		-1		$\boxed{12}$
2	1		$\boxed{10}$	
		$\boxed{9}$		
3	$\boxed{10}$			

$$y(4) = 10 + 1(9) + 1\frac{(1+1)}{2}(10) + \frac{6}{6}\times 12.$$

$$= 10+9+10+12$$

$$= 41.$$

To find the polynomials using forward formula.

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots$$

$$u = \frac{x-x_0}{h} = \frac{x-0}{1} = x \quad \therefore \boxed{u = x}$$

$$y = 1 + x.1 + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{6}\times 12$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$= 2x^3 - 7x^2 + 6x + 1.$$

Put $y = 0$

$$x = \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13-0)(-13-3)(-13-18)}(34)$$

$$+ \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$

$$\boxed{x = 37.23}$$

4. Using Newtons forward interpolation formula find the Cubic polynomial which takes the following values.

x :	0	1	2	3
f(x) :	1	2	1	10

Evaluate f(4) using Newtons backward formula.

Solution:

Here $x = 4$

$$v = \frac{x - x_n}{h} = \frac{4-3}{1} = 1 \quad \therefore \boxed{v = 1}$$

Formula:

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \ldots$$

5. Find f(x) as a polynomial in x for the following data by Newton's divided difference formula.

x :	-4	-1	0	2	5
f(x) :	1245	33	5	9	1335.

Table Calculation:

x	f(x)	Δf(x)	Δ²f(x)	Δ³f(x)	Δ⁴f(x)
-4	1245				
		$\frac{33-1245}{-1-(-4)} = -404$			
-1	33		$\frac{-28-(-404)}{0-(-4)} = 94$		
				$\frac{10-94}{2-(-4)} = -14$	
0	5	$\frac{5-33}{0-(-1)} = -28$			$\frac{13+14}{5-(-4)} = 3$
			$\frac{2-(-28)}{2-(-1)} = 10$		
2	9	$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5-(-1)} = 13.$	
			$\frac{442-2}{5-0} = 88$		
5	1335	$\frac{1335-9}{5-2} = 442$			

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2)$$

$$+ (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94)$$

$$+ (x+4)(x+1)(x)(-14)$$

$$+ (x+4)(x+1)(x-2)(3).$$

$$= 3x^4 + x^3 - 14x + 5$$

6. Find F(8) by Newton's divided difference formula

x :	4	5	7	10	11	13
f(x) :	48	100	294	900	1210	2028

x	f(x)	Δf(x)	Δ²f(x)	Δ³f(x)	Δ⁴f(x)	Δ⁵f(x)
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2)$$

$$= 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)$$

Put $x = 8$.

$$f(8) = 448.$$

7. Find the Cubic Spline for following data given below:

x :	0	1	2
y :	0	1	0

To Find $y(0.5)$ and $y'(1)$.

Solution:

Here $h=1$; $n=2$; $i = 1,2,\ldots n-1$

$i = 1,2,\ldots 2-1$

$i = 1,2,\ldots 1$

Assume that $M_0 = M_2 = 0$.

Where

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$$(0) + 4M_1 + (0) = 6[0 - 2(1) + 0]$$

$$4M_1 = 6[-2+0]$$

$$4M_1 = -12$$

$$\boxed{M_1 = -3}$$

Let $x_{i-1} \le x \le x_i$

$x_0 \le x \le x_1$

$0 \le x \le 1$

By, the Cubic Spline Formula,

$$S(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[(x_i - x)(y_{i-1} - \tfrac{h^2}{6}(M_{i-1}))] + \frac{1}{h}[(x - x_{i-1})(y_i - \tfrac{h^2}{6}M_i)]$$

Put $i=1$,

$$y(x) = S(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + 1[(x_1 - x)(y_0 - \tfrac{1}{6}(M_0))] + 1[(x - x_0)(y_1 - \tfrac{1}{6}M_1)]$$

$$= \frac{1}{6}[(1-x)^3(0) + (x-0)^3(-3)] + [(1-x)(0 - \tfrac{1}{6}(0))] + [(x-0)(1 - \tfrac{1}{6}(-3))]$$

$$= \frac{1}{6}[0 - 3x^3] + [0] + [(x)(1.5)]$$

$$\boxed{S(x) = -0.5x^3 + 1.5x}$$

To find $y(0.5)$ and $y'(1)$:

Here $y(x) = S(x) = -0.5x^3 + 1.5x$

$$y(0.5) = -0.5(0.5)^3 + 1.5(0.5)$$

$$= -0.0625 + 0.75$$

$$\boxed{y(0.5) = 0.6875}$$

Here $y(x) = -0.5x^3 + 1.5x$

$$y'(x) = -(3)\times 0.5x^2 + 1.5$$

$$y'(1) = -3\times 0.5(1)^2 + 1.5$$

$$= -1.5 + 1.5$$

$$\boxed{y'(1) = 0}$$

By Using Cubic Spline formula,

$$S(x) = \frac{1}{6h}[(x_i - x)^3 (M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[(x_i - x)(y_{i-1}) - \tfrac{h^2}{6}M_{i-1})] + \frac{1}{h}[(x - x_{i-1})(y_i - \tfrac{h^2}{6}M_i)]$$

Put $i=1$ (i.e) $x_{i-1} \le x \le x_i$ $(-1 \le x \le 0)$

$$S(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + \frac{1}{1}[(x_1 - x)(y_0 - \tfrac{1}{6}M_0)] + [(x - x_0)(y_1 - \tfrac{1}{6}M_0)]$$

$$= \frac{1}{6}[(0-x)^3\, 10 + (x+1)^3(-2)] + [(0-x)(1 - \tfrac{1}{6}(10))] + 1[(x+1)(0 - \tfrac{1}{6}(-2))]$$

$$= \frac{2}{6}[-5x^3 - (x+1)^3] + [(-x)(-\tfrac{2}{3})] + [(x+1)(\tfrac{1}{3})]$$

$$= \frac{1}{3}[-5x^3 - x^3 - 3x^2 - 3x - 1] + \frac{2}{3}x + \frac{1}{3}x + \frac{1}{3}$$

$$= \frac{1}{3}[-6x^3 - 3x^2 - 3x - 1] + \frac{2}{3}x + \frac{1}{3}x + \frac{1}{3}$$

$$= -2x^3 - x^2 - \cancel{x} - \frac{1}{3} + \cancel{x} + \frac{1}{3}$$

$$= -2x^3 - x^2$$

$\therefore S(x) = -2x^3 - x^2$ at $(-1 \le x \le 0)$

Thus

$$\boxed{\begin{aligned} y(x) = S(x) &= -0.5x^3 + 1.5x \\ y(0.5) &= 0.6875 \\ y'(1) &= 0 \end{aligned}}$$

8. Find the Cubic Spline for the following data with $M_0 = 10$, $M_2 = 10$.

x :	-1	0	1
y :	1	0	1

Solution:

Here $h=1$; $n=2$ $i = 1,2,\ldots n-1$

$i = 1,2,\ldots 1$

Here $M_0 = M_2 = 10$.

Where

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

Put $i=1$

$$M_0 + 4M_1 + M_2 = \frac{6}{1}[y_0 - 2y_1 + y_2]$$

$$10 + 4M_1 + 10 = 6[1 - 0 + 1]$$

$$20 + 4M_1 = 12$$

$$4M_1 = 12 - 20 = -8$$

$$\boxed{M_1 = -2}$$

Put $i=2$ (ie) $x_{i-1} \le x \le x_i$ $(0 \le x \le 1)$.

$$S(x) = \tfrac{1}{6}\left[(x_2-x)^3 M_1 + (x-x_1)^3 M_2\right]$$
$$+\left[(x_2-x)(y_1 - \tfrac{1}{6}M_1)\right] + \left[(x-x_1)(y_2 - \tfrac{1}{6}M_2)\right]$$
$$= \tfrac{1}{6}\left[(1-x)^3(-2) + (x-0)^3 10\right] + \left[(1-x)(0-\tfrac{1}{6}(-2))\right]$$
$$+\left[(x-0)(1-\tfrac{1}{6}(10))\right]$$
$$= \tfrac{2}{6}\left[-(1-x)^3 + 5x^3\right] + (1-x)(\tfrac{1}{3}) + x(1-\tfrac{5}{3})$$
$$= \tfrac{1}{3}\left[5x^3 + x^3 - 3x^2 + 3x - 1\right] + \tfrac{1}{3} - \tfrac{x}{3} - \tfrac{2}{3}x$$
$$= \tfrac{1}{3}\left[6x^3 - 3x^2 + 3x - 1\right] + \tfrac{1}{3} - x$$
$$= 2x^3 - x^2 + \cancel{x} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{x}$$
$$= 2x^3 - x^2$$

$\therefore S(x) = 2x^3 - x^2$ at $0 \le x \le 1$

Thus
$$\boxed{\begin{array}{l} S_1(x) = -2x^3 - x^2 \quad (-1 \le x \le 0) \\ S_2(x) = 2x^3 - x^2 \quad (0 \le x \le 1) \end{array}}$$